Recall: The dimension of a vector space V over a field $\mathbb{F}$ is the cardinality of a basis.

We showed that dimension is well-defined.

Corollary: (subspaces)

If V is a vector space over $\mathbb{F}$ and W is a subspace of V, then if we let $\dim_{\mathbb{F}}(V)$ denote the dimension,

$$\boxed{\dim_{\mathbb{F}}(W) \leq \dim_{\mathbb{F}}(V)}$$

Proof: Let B be a basis for W over $\mathbb{F}$. Then the elements of B are linearly independent. Then B is contained in a maximal linearly independent subset, call it B_0, of V. B_0 is a basis for V, so

$$|B| \leq |B_0|$$

But $|B| = \dim_{\mathbb{F}}(W)$

and $|B_0| = \dim_{\mathbb{F}}(V)$,

so $\dim_{\mathbb{F}}(W) \leq \dim_{\mathbb{F}}(V)$.

$\square$

Proposition: (finite dimensions)

If V is a finite-dimensional vector space over $\mathbb{F}$ ($\dim_{\mathbb{F}}(V)$ is finite), then if W is any proper subspace of V,

$$\boxed{\dim_{\mathbb{F}}(W) < \dim_{\mathbb{F}}(V)}$$

Proof: By the previous corollary, we know

$$\dim_{\mathbb{F}}(W) \leq \dim_{\mathbb{F}}(V).$$

Assume, by way of contradiction, that

$$\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(V) = n \in \mathbb{N}.$$

Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for W.

We know the elements in B are linearly independent. Since W is proper, $\exists$ $v \in V \setminus W$.

Then $v \notin \mathrm{span}(B)$.

Therefore, the set

$$B_0 = \{b_1, b_2, \ldots, b_n, v\}$$

is linearly independent in V. This implies $\dim_{\mathbb{F}}(V) \geq n+1$. But $\dim_{\mathbb{F}}(V) = n$ by assumption, contradiction.

Therefore, $\dim_{\mathbb{F}}(W) < \dim_{\mathbb{F}}(V)$.

$\square$

Example 1: (infinite dimensions)

If $V = c_0$, the space of all complex sequences converging to zero, as a vector space over $\mathbb{C}$, the subspace

$$W = \{ (a_i)_{i=1}^{\infty} \in c_0 \mid a_1 = 0 \}$$

has $\dim_{\mathbb{F}}(W) = \dim_{\mathbb{F}}(c_0)$.

In general, an infinite dimensional vector space always admits a proper subspace with the same dimension:

If $B = \{b_i\}_{i \in I}$ with I infinite, choose

$K \subsetneq I, \quad |K| = |I|$

Then let

$$W = \operatorname{span}\left(\{b_k\}_{k \in K}\right).$$

W is a proper subspace of V with the same dimension as V.

Chapter 6

Definition: (norm) Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ such that $\forall\, x, y, z \in V$,

1) $\|x\| = 0$ if and only if

$x = 0_V$.

2) $\|\alpha x\| = |\alpha| \, \|x\|$

$\forall \ \alpha \in \mathbb{R}$ or $\mathbb{C}$

3) $\|x-y\| \leq \|x-z\| + \|z-y\|$

(triangle inequality)

Example 2: (2-norm)

Consider $V = \mathbb{C}^2$ as a vector space over $\mathbb{C}$.

For $w = (w_1, w_2) \in \mathbb{C}^2$, define the 2-norm of w to be

$$\boxed{\|w\|_2 = \sqrt{|w_1|^2 + |w_2|^2}}$$

Why is this a norm?

1) $\|x\|_2 = 0$ if and only if

$$\underline{x = (0, 0)}.$$

Suppose $\|x\|_2 = 0$. Then

if $x = (x_1, x_2)$,

$$\sqrt{|x_1|^2 + |x_2|^2} = 0.$$

But then $|x_1| = |x_2| = 0$,

so $x = (0, 0)$.

$$\|(0,0)\|_2 = \sqrt{0^2+0^2} = 0.$$

2) $\|\alpha x\|_2 = |\alpha|\,\|x\|_2$

$\forall\ x \in V,\ \alpha \in \mathbb{C}$

If $x = (x_1, x_2)$,

$$\|\alpha x\|_2 = \sqrt{|\alpha x_1|^2 + |\alpha x_2|^2}$$

$$= \sqrt{|\alpha|^2(|x_1|^2 + |x_2|^2)}$$

$$= |\alpha| \sqrt{|x_1|^2 + |x_2|^2}$$

$$= |\alpha|\ \|x\|_2$$

3) <u>Triangle Inequality</u>

Take $x = (x_1, x_2)$,

$y = (y_1, y_2)$, $z = (z_1, z_2)$,

elements in $\mathbb{C}^2$. Show

$$\|x-y\|_2 \leq \|x-z\|_2 + \|z-y\|_2.$$

Square both sides to

get

$$\left(|(x_1-y_1)|^2+|x_2-y_2|^2\right)$$

$$\leq |(x_1-z_1)|^2+|(x_2-z_2)|^2$$

$$+2\,\|x-y\|_2\cdot\|y-z\|_2$$

$$+|(z_1-y_1)|^2+|(z_2-y_2)|^2$$

Now again using the triangle inequality for complex numbers, we obtain

$$|x_1 - y_1|^2 + |x_2 - y_2|^2$$

$$\leq \left(|x_1 - z_1| + |z_1 - y_1|\right)^2$$

$$+ \left(|x_2 - z_2| + |z_2 - y_2|\right)^2$$

$$= |x_1 - z_1|^2 + 2|x_1 - z_1||z_1 - y_1|$$

$$+ |z_1 - y_1|^2 + |x_2 - z_2|^2$$

$$+ 2|x_2 - z_2||z_2 - y_2|$$

$$+ |z_2 - y_2|^2.$$

Comparing this with our original inequality squared (look back a few pages!), we see it suffices to show

$$\boxed{|x_1 - z_1||z_1 - y_1| + |x_2 - y_2||z_2 - y_2| \leq \|x - z\|_2 \|z - y\|_2}$$

Squaring

$$|x_1 - z_1||z_1 - y_1| + |x_2 - y_2||z_2 - y_2|,$$

we obtain

$$|x_1 - z_1|^2 |z_1 - y_1|^2$$

$$+ 2|x_1 - y_1||x_2 - y_2||z_1 - y_1||z_2 - y_2|$$

$$+ |x_2 - y_2|^2 |z_2 - y_2|^2$$

Squaring

$$\|x - z\|_2 \cdot \|z - y\|_2$$

we obtain

$$|x_1 - z_1|^2 |z_1 - y_1|^2$$
$$+ |x_2 - z_2|^2 |z_2 - y_2|^2$$
$$+ |x_2 - z_2|^2 |z_1 - y_1|^2$$
$$+ |x_1 - z_1|^2 |z_1 - y_1|^2 .$$

Again comparing terms, we see it is sufficient to prove

$$2|x_1-y_1||x_2-y_2||z_1-y_1||z_2-y_2|$$
$$\leq |x_2-z_2|^2|z_1-y_1|^2$$
$$+|x_1-z_1|^2|z_2-y_2|^2.$$

But subtracting and collecting terms, this is

$$0 \leq |x_2 - z_2|^2 |z_2 - y_2|^2 + |x_1 - z_1|^2 |z_1 - y_1|^2$$

$$- 2|x_1 - y_1||x_2 - y_2||z_1 - y_1||z_2 - y_2|$$

But this last quantity is equal to

$$\left(|x_1 - z_1||z_2 - y_2| - |x_1 - z_1||z_1 - y_1|\right)^2,$$

which is always non-negative. Therefore the triangle inequality follows.